Statistical approach to nonhyperbolic chaotic systems

D. Alonso,* D. MacKernan, P. Gaspard, and G. Nicolis
Center for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles, Campus Plaine Code Postal 231, B-1050 Bruxelles, Belgium
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Spectral properties of the evolution operator for probability densities are obtained for unimodal maps for which all periodic orbits are unstable, and the Lyapunov exponent calculated from the first iterate of the critical point converges to a positive constant. The method is applied to the logistic map both for parameter values at which finite Markov partitions can be found as well as for more typical parameter values. A universal behavior is found for the spectral gap in the period-doubling inverse cascade of chaotic band-merging bifurcations. Full agreement with numerical simulation is obtained. [S1063-651X(96)07009-2]

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I. INTRODUCTION

The importance of the period-doubling route to chaos is by now well established in such diverse fields as fluid dynamics, optics, chemistry, and biology [1–5]. In this context, a most remarkable feature is that one-dimensional discrete time dynamics governed by a unimodal mapping is often observed when return maps are constructed from experimental time series pertaining to a single variable, or from mathematical models of multivariate continuous-time dynamical systems [6]. As a result, many qualitative and even quantitative features of wide classes of systems can be captured through the analysis of simple dynamical systems, such as the logistic map.

The properties of unimodal maps in the range in which stable periodic orbits exist and no chaos is present is by now a practically solved problem. On the other hand, the behavior of such systems after the onset of chaos, where one observes (with the exception of periodic windows) stochasticlike motion is less understood despite intense investigations. It is widely recognized that in this range the most appropriate method of description is the statistical one, based on the evolution equation for probability densities known as the Frobenius-Perron equation. The problem is thus reduced to finding a workable representation of the probability density in terms of the spectral properties of the corresponding Frobenius-Perron operator $\hat{P}$. A number of results in this direction have been reported essentially for hyperbolic maps and, especially, for piecewise linear maps [4,7–9]. For such systems, the ζ-function formalism has also been widely used to obtain decay rates of time correlation functions. Much less understood, both as far as the Frobenius-Perron operator and the ζ-function formalism are concerned, is the behavior of nonhyperbolic chaotic systems, which include the important class of unimodal maps, and in particular the logistic map in the chaotic regime. Our objective in the present paper is to show that the statistical behavior of such systems, including invariant densities, time-correlation functions, and decay rates can be evaluated through an explicit matrix representation of the evolution operator $\hat{P}$.

The chief difficulty posed by the logistic map and several other experimentally obtained first-return maps is due to the existence of a quadratic extremum. The corresponding $\hat{P}$, when applied recursively on initially analytic functions, produces singularities at all the iterates of the extremum. A method of treating these singularities is vital to any understanding of the statistical behavior of the system. Fortunately, these singularities are integrable if the Lyapunov exponent, when evaluated from the first iterate of the extremum, converges to a positive constant. The first objective of this paper is the creation of basis sets of orthogonal functions capable of handling such singularities, as exposed in Sec. II. This then permits one to obtain a matrix representation $\mathcal{W}$ of $\hat{P}$, which, typically, can be truncated when all periodic orbits are unstable. The eigenvectors and eigenvalues of $\mathcal{W}$ define, through the basis, a class of eigenfunctions and eigenvalues of $\hat{P}$. One can thus calculate time-correlation functions, spectra, and invariant densities. In Sec. III, the procedure will finally be illustrated on a few representative examples. In Sec. IV, we obtain a universal formula for the spectral gap as a function of the control parameter. This relation allows one to predict how quickly time correlation functions decay to their asymptotic values as the onset of chaos is approached from above.

II. MATRIX REPRESENTATION OF THE FROBENIUS-PERRON OPERATOR

A. The Frobenius-Perron operator and correlation functions

Let us consider a one-dimensional dynamical system on the unit interval, $x_{t+1}=f(x_t)$, an example of which is the logistic map, $f_r(x)=rx(1-x)$, wherein the control parameter $r\geqslant 0$. As is well known, this example exhibits chaos at values of $r\geqslant r_c=3.56994\ldots$. The corresponding Frobenius-Perron operator $\hat{P}$ is defined through the equation [10]

$$\rho_{t+1}(x)=\hat{P}\rho_t(x)=\sum_{\alpha}\frac{\rho_t[f_\alpha^{-1}(x)]}{|f'[f_\alpha^{-1}(x)]|}, \tag{1}$$

where $f_\alpha^{-1}(x)$ denotes the local inverse branches of f. It is

*Present address: Department of Fundamental and Experimental Physics, Universidad de La Laguna, 38203 La Laguna, Spain understood that if $f_\alpha^{-1}(x)$ is not defined for a point x, which happens when a given trajectory is not physically realizable, then the corresponding term in the sum Eq. (1) is set equal to zero at this point. The Frobenius-Perron operator is useful not only in that it evolves probability densities, but also because it can be used to evaluate time-correlation functions, provided that one knows the asymptotic state of the system [typically the invariant density $\rho_{\rm eq}(x)$]. The time-correlation function $C_{AB}(\tau)$ of two measurable functions $A(x)$ and $B(x)$ (also referred to as observables) usually calculated from a long time series data, can thus be alternatively given in terms of [11]

$$C_{AB}(\tau)=\int_0^1 A(x)[\hat{P}^\tau(B\rho_{\rm eq})](x)dx-\langle A\rangle_{\rm eq}\langle B\rangle_{\rm eq}. \quad (2)$$

Clearly, a knowledge of the properties of $\hat{P}$ when acting on functions of the type $B(x)\rho_{\rm eq}(x)$ is sufficient to calculate time-correlation functions.

B. Formation of singularities in nonhyperbolic maps

Before proceeding to the study of the spectral properties of the Frobenius-Perron operator, we first discuss the formation of singularities when $\hat{P}$ is applied successively on a smooth initial probability density $\rho_0(x)$. We suppose that the map $f(x)$ has a single quadratic critical point x_c, such that $f'(x_c)=0$, $f''(x_c)\neq 0$ [with the notation $f'(x)=df/dx$]. For the logistic map, the critical point is located at $x_c=1/2$. Let us denote the iterates of the critical point $f(x_c), f[f(x_c)], \ldots, f^{t+1}(x_c), \ldots$ by the set $\{x_{ct}\}$, with $t=0,1,2,\ldots$, and consider first the case where they are finite in number (N_c+1). This happens when the critical point falls in a finite number of iterations onto a periodic orbit. We observe that the probability density $\hat{P}^t\rho_0$ develops an extra new singularity at each iteration until the last point of the periodic orbit is finally visited after N_c iterations. Because the critical point is quadratic and provided that the relative stability factor of the sequence, defined by

$$\Lambda=\prod_{t=0}^{N_c}|f'(x_{ct})|>1, \quad (3)$$

is greater than 1, we can conclude that all these singularities are of square root type. More generally, the evolved probability density behaves as

$$\hat{P}^t\rho_0=\sum_{n=0}^{\infty}\alpha_n(t)|x-x_{ct}|^{(n-1)/2}, \quad (4)$$

on one side or the other of the points $\{x_{ct}\}$.

For convenience, let us reorder the set $\{x_{ct}\}$ according to increasing values and denote the new set by $\{a_i\}_{i=0}^{N_c}$. The set can be used to partition the interval $[a_0,a_{N_c}]$ into N_c cells $C_i=[a_{i-1},a_i]$ with $i=1,2,\ldots,N_c$. In the literature, this partition is often referred to as the minimal Markov partition [12]. To simplify the discussion without loss of generality, we shall only consider the nontransient dynamics, which is restricted to the interval $[a_0,a_{N_c}]$.

C. The orthogonal basis

The method we propose uses a special basis of orthogonal functions of the type

$$\phi_{im}(x)=s_i(x)T_m[\eta_i(x)]\chi_i(x), \quad (5)$$

where $\{T_n(z)\}$, $n=0,1,2,\ldots$ is any basis of polynomial functions (such as the Chebyshev polynomials, for example [13]) with support on the unit interval and orthogonal with respect to a weighting function $w(z)$,

$$\int_0^1 dz\ \ w(z)T_m(z)T_n(z)=\gamma_m\delta_{mn}. \quad (6)$$

$\chi_i(x)$ is the characteristic function of the cell C_i (i.e., equals one if $x\in C_i$ and zero otherwise). The functions $s_i(x)$ are chosen in such a way that the basis functions Eq. (5) have the same type of square root singularities as the evolved probability densitites (4). An example of such a function is given by

$$s_i(x)=\frac{1}{\pi\sqrt{(x-a_{i-1})(a_i-x)}}, \quad (7)$$

where the factor $(1/\pi)$ is adopted for normalization.

In our method, the functions $\eta_i(x)$ in Eq. (5) are related to the functions $s_i(x)$ by the following argument. We assume that the basis functions (5) also obey an orthogonality relation

$$\int_0^1 dx\ \ \tilde{w}(x)\phi_{im}(x)\phi_{jn}(x)=\gamma_m\delta_{mn}\delta_{ij}, \quad (8)$$

with respect to another weighting factor, $\tilde{w}(x)=\sum_{i=1}^{N_c}\tilde{w}_i(x)\chi_i(x)$, than in Eq. (6). Inserting the definition (5) of the basis functions in Eq. (8), we deduce the relation

$$\int_{a_{i-1}}^{a_i} dx\ \ \tilde{w}_i(x)[s_i(x)]^2T_m[\eta_i(x)]T_n[\eta_i(x)]=\gamma_m\delta_{mn}. \quad (9)$$

After the change of variable $z=\eta_i(x)$, we recover the previous orthogonality relation (6) under the conditions

$$\eta_i(a_{i-1})=0,\quad \eta_i(a_i)=1, \quad (10)$$

$$s_i(x)=\eta_i'(x), \quad (11)$$

$$\tilde{w}_i(x)=\frac{w[\eta_i(x)]}{\eta_i'(x)} \quad (12)$$

for $x\in C_i$. We assumed here that the conditions (11) and (12) are satisfied separately, which is specific to our method. The conditions (10)–(12) allow us to determine the functions $\eta_i(x)$. The particular choice (7) of $s_i(x)$ yields

$$\eta_i(x)=\frac{2}{\pi}\arcsin\sqrt{\frac{x-a_{i-1}}{a_i-a_{i-1}}}, \quad (13)$$

which completes the construction of our basis. Since our basis functions are tailored to the intrinsic square-root type singularities of the system, exhibited either by probability densities or their derivatives [see Eq. (4)], we obtain rapid convergence. Moreover, one can easily show that many of the completeness properties of the basis (6) are shared by the new basis (5).

D. The matrix representation

A matrix representation of $\hat{P}$ can now be obtained,

$$(\mathcal{W})_{imjn}=\frac{1}{\gamma_m}\int_0^1 dx\ \tilde{w}(x)\phi_{im}(x)(\hat{P}\phi_{jn})(x), \quad (14)$$

by simply regrouping the indices so that $(\mathcal{W})_{imjn}=M_{kl}$, where $k=mN_c+i$ and $l=nN_c+j$, for instance. Eigenvalues and eigenfunctions (such that $\hat{P}\psi_\nu=z_\nu\psi_\nu$) can then be found by truncating M_{kl} at a sufficiently high order (corresponding to $m=n\leqslant N_p$, where typically $N_p\leqslant 10$) as for piecewise-linear maps [14,15]. In particular, the invariant density—when unique—corresponds to the eigenvector of $\mathcal{W}$ with unit eigenvalue.

Let $\rho_0(x)$ denote an initial density. In the correlation-function formalism, Eq. (2), the role of such an initial state is played by $B(x)\rho_{\text{eq}}(x)$. Then,

$$\rho_0(x)=\sum_{i=1}^{N_c}\sum_{m=0}^{N_p} c_{in}\phi_{im}(x), \quad (15)$$

and

$$\hat{P}^t\rho_0(x)=\sum_{i,j=1}^{N_c}\sum_{m,n=0}^{N_p}\phi_{im}(x)(\mathcal{W}^t)_{imjn}c_{jn}, \quad (16)$$

so that correlation functions such as (2) can be easily evaluated.

So far, we assumed that the iterates of the critical point form a finite set. More generally, the set is countably infinite, which would appear problematic. In fact, the relative magnitude of the singularities rapidly becomes negligible after several iterations. Thus the Frobenius-Perron operator can be well approximated using only a finite number of singularities and by increasing this number to improve the approximation. In practice, spectra converge to stable values once Λ [see Eq. (3)] approaches its limiting value.

III. EXAMPLES

A. The logistic map when $r=4$

Our method of treating the singularities is equivalent to a conjugacy only when two square-root type singularities can appear in evolving probability densities of the system (even if the initial density is smooth), as in the logistic map for $r=4$. In this case the singularities only occur at zero and one, so that $a_0=0$, $a_1=1$, and $N_c=1$. The conjugacy [7] to the tent map $g(y)=1-|2y-1|$ is established by $g=\eta\circ f\circ\eta^{-1}$ with

$$\eta(x)=\frac{2}{\pi}\arcsin\sqrt{x},\quad s(x)=\eta'(x)=\frac{1}{\pi\sqrt{x(1-x)}}. \quad (17)$$

The Bernoulli polynomials $\{B_{2\nu}(y/2)\}$ [16] are eigenfunctions of the tent map with corresponding eigenvalues $z_\nu=2^{-2\nu}$ for $\nu=0,1,2,\ldots$ [8]. It is a simple matter to verify that the functions

$$\psi_\nu(x)=\eta'(x)B_{2\nu}[\eta(x)/2]=\frac{1}{\pi\sqrt{x(1-x)}}B_{2\nu}\left(\frac{1}{\pi}\arcsin\sqrt{x}\right) \quad (18)$$

are eigenfunctions of the logistic map with the same spectrum as the tent map. In fact, as $\eta'(x)$ is the invariant density of the logistic map for $r=4$, our choice of basis is essentially equivalent to a local equilibrium hypothesis, where the densities are equal to continuous functions times the equilibrium density. A complete spectral decomposition can also be obtained.

B. The logistic map when $r<4$

We now come to the general case of parameter values of r below $r=4$. To realize the nature of the problem, consider, for instance, $r=3.95$, for which numerical simulations reveal the existence of a chaotic attractor. For this value, no finite minimal Markov partition can be found for at least the first 400 iterations of the extremum. However, one expects that there are values $\tilde{r}$ arbitrarily close to r such that a finite minimal Markov partition exists, that is, if the extremum falls onto a periodic orbit after a certain number of iterates. For $r=3.95$, one finds that the itinerary of $\frac{1}{2}$ almost repeats itself after 22 and 46 iterations, respectively. The corresponding values of $\tilde{r}$ at which a finite Markov partition exists are 3.950 000 101 121 850 5454 and 3.949 999 999 999 975 4036. Alternatively, for $r=3.95$, one can construct a partition with the first N_c+1 iterates of the extremum and make an approximation of the type (16) using a finite basis $\{\phi_{im}(x)\}$ [see Eq. (5)]. One then seeks a value of N_c and N_p such that one observes convergence in the quantities of interest as N_c and N_p are increased. In this way, we have calculated the time-correlation functions, invariant

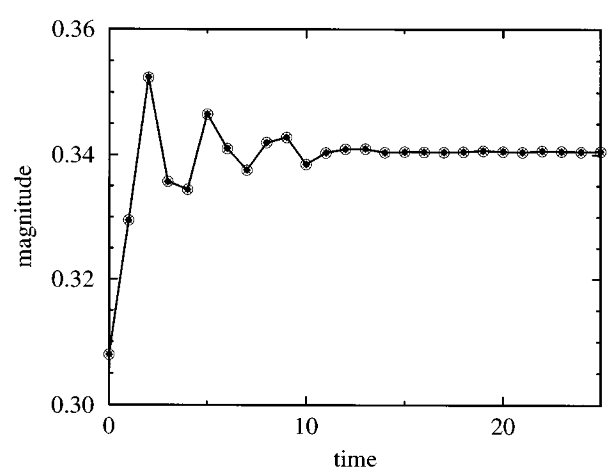

FIG. 1. Time-correlation function $\langle x_0x_t\rangle_{\text{eq}}$ for the logistic map $x_{t+1}=rx_t(1-x_t)$ obtained numerically for $r=3.95$ (solid line) and with the method described in this paper for $r=3.95$ (crosses), $r=3.950\,000\,101\,121\,850\,5454$ (circles), and $r=3.949\,999\,999\,999\,975\,4036$ (diamonds).

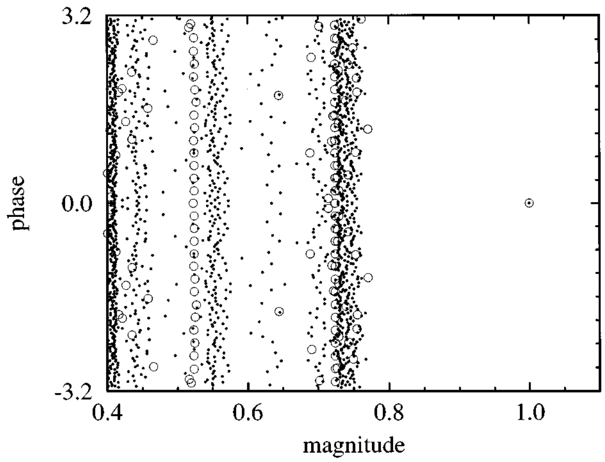

FIG. 2. Spectra of eigenvalues of the Frobenius-Perron operator (numerically given by the eigenvalues of the matrix $\mathcal{W}$) associated with the logistic map for $r=3.95$ (dots) and $r=3.949\,999\,999\,999\,975\,4036$ (circles).

densities, and spectra of the logistic map for the three mentioned values of r. As a check, we have also calculated the corresponding time-correlation functions through a long time series. Figure 1 shows that the time-correlation functions of the three systems are virtually indistinguishable. In contrast, the spectra show a sensitivity to parametric perturbations although they have qualitatively the same structure (see Fig. 2).

IV. UNIVERSALITY IN THE BAND-MERGING CASCADE

The method also allows one to study the spectral properties of the Frobenius-Perron operator in the inverse cascade of band-merging bifurcations $\{r_n\}$ accumulating from above at the onset of chaos as $(r_n-r_c)\sim\delta^{-n}$, where $\delta=4.6692\ldots$ is the Feigenbaum constant [4,5]. As a first result, the number of eigenvalues on the unit circle is equal to 2^n when 2^n bands merge, leading to period-2^n chaos, which is the spectral signature of an ergodic but nonmixing dynamics [17]. Moreover, the rest of the spectrum turns out to be separated from the unit circle by a gap that shrinks as the nonchaotic regime is approached as

$$\Delta_{\rm gap}=\mathrm{Min}_{\nu=2^n+1,2^n+2,\ldots}\{1-|z_\nu|\}\sim(r_n-r_c)^\tau,$$

$$\text{with}\quad \tau=\frac{\ln 2}{\ln\delta}=0.449\,8069\ldots\ . \tag{19}$$

The universal exponent is determined by noting that the gap is of the order of the average Lyapunov exponent, which is known to scale as $\lambda\sim|r-r_c|^\tau$ [18]. We carried out a numerical verification of this prediction in the logistic map where the spectrum has been calculated down to the merging of 64 chaotic bands, giving an exponent $\tau_{\rm num}\simeq0.44$ in good agreement with Eq. (19). This result shows that the spectrum of the Frobenius-Perron operator displays universal behaviors in the period-doubling cascade.

V. CONCLUDING REMARKS

In conclusion, the method developed in this paper constitutes a first step towards the statistical study of experimental systems, most naturally modeled by nonhyperbolic chaotic attractors. Explicit applications to unimodal one-dimensional maps arising from the far-from-equilibrium Belousov-Zhabotinskii chemical reaction [19] and electrochemical oscillators [20] have been outlined. The method can be readily extended to other maps with several critical points, which are either quadratic or degenerate.

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